Some Liouville theorems and applications

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Dedicated to Haim Brezis with high respect and friendship

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Abstract

We give exposition of a Liouville theorem established in [6] which is a novel extension of the classical Liouville theorem for harmonic functions. To illustrate some ideas of the proof of the Liouville theorem, we present a new proof of the classical Liouville theorem for harmonic functions. Applications of the Liouville theorem, as well as that of earlier ones in [5], can be found in [6, 7] and [9].

The Laplacian operator Δ is invariant under rigid motions: For any function u on $\mathbb{R}^n$ and for any rigid motion $T:\mathbb{R}^n\to\mathbb{R}^n$,

$$\Delta(u\circ T)=(\Delta u)\circ T.$$

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The following theorem is classical:

$$u \in C^2, \quad \Delta u = 0 \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{ constant.} \tag{1}$$

In this note we present a Liouville theorem in [6] which is a fully nonlinear version of the classical Liouville theorem (1).

Let u be a positive function in $\mathbb{R}^n$, and let $\psi : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ be a Möbius transformation, i.e. a transformation generated by translations, multiplications by nonzero constants and the inversion $x \to x/|x|^2$. Set

$$u_\psi := |J_\psi|^{\frac{n-2}{2n}} (u \circ \psi),$$

where J_ψ is the Jacobian of ψ.

It is proved in [3] that an operator $H(u, \nabla u, \nabla^2 u)$ is conformally invariant, i.e.

$$H(u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv H(u, \nabla u, \nabla^2 u) \circ \psi \text{ holds for all positive } u \text{ and all Möbius } \psi,$$

if and only if H is of the form

$$H(u, \nabla u, \nabla^2 u) \equiv f(\lambda(A^u))$$

where

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$

I is the $n \times n$ identity matrix, $\lambda(A^u) = (\lambda_1(A^u), \cdots, \lambda_n(A^u))$ denotes the eigenvalues of A^u, and f is a function which is symmetric in $\lambda = (\lambda_1, \cdots, \lambda_n)$.

Due to the above characterizing conformal invariance property, A^u has been called in the literature the conformal Hessian of u. Since

$$\sum_{i=1}^n \lambda_i(A^u) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u,$$

Liouville theorem (1) is equivalent to

$$u \in C^2, \quad \lambda(A^u) \in \partial \Gamma_1 \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{ constant,} \tag{2}$$

where

$$\Gamma_1 := \{\lambda \mid \sum_{i=1}^n \lambda_i > 0\}.$$

Let

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex symmetric cone with vertex at the origin} \tag{3}$$

satisfying

$$\Gamma_n := \{\lambda \mid \lambda_i > 0, 1 \le i \le n\} \subset \Gamma \subset \Gamma_1. \tag{4}$$

Examples of such Γ include those given by elementary symmetric functions. For $1 \le k \le n$, let

$$\sigma_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the $k-$th elementary symmetric function and let $\Gamma_k := \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \cdots, \sigma_k(\lambda) > 0\}$, which is equal to the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone Γ_n, satisfies (3) and (4).

For an open subset Ω of $\mathbb{R}^n$, consider

$$\lambda(A^u) \in \partial\Gamma, \qquad \text{in } \Omega. \tag{5}$$

The following definition of viscosity super and sub solutions of (5) has been given in [6].

Definition 1 *A positive continuous function u in Ω is a viscosity subsolution [resp. supersolution] of (5) when the following holds: if $x_0 \in \Omega$, $\psi \in C^2(\Omega)$, $(u-\psi)(x_0) = 0$ and $u - \psi \le 0$ near x_0 then*

$$\lambda(A^\psi(x_0)) \in \mathbb{R}^n \setminus \Gamma.$$

[resp. if $(u-\psi)(x_0) = 0$ and $u - \psi \ge 0$ near x_0 then $\lambda(A^\psi(x_0)) \in \overline{\Gamma}$].

We say that u is a viscosity solution of (5) if it is both a viscosity supersolution and a viscosity subsolution.

Remark 1 *If a positive u in $C^{1,1}(\Omega)$ satisfies $\lambda(A^u) \in \partial\Gamma$ a.e. in Ω, then it is a viscosity solution of (5).*

Here is the Liouville theorem.

Theorem 1 *([6]) For $n \ge 3$, let Γ satisfy (3) and (4), and let u be a positive locally Lipschitz viscosity solution of*

$$\lambda(A^u) \in \partial\Gamma \qquad \text{in } \mathbb{R}^n. \tag{6}$$

Then $u \equiv u(0)$ in $\mathbb{R}^n$.

Remark 2 *It was proved by Chang, Gursky and Yang in [1] that positive* $C^{1,1}(\mathbb{R}^4)$ *solutions to* $\lambda(A^u) \in \partial\Gamma_2$ *are constants. Aobing Li proved in [2] that positive* $C^{1,1}(\mathbb{R}^3)$ *solutions to* $\lambda(A^u) \in \partial\Gamma_2$ *are constants, and, for all* k *and* n*, positive* $C^3(\mathbb{R}^n)$ *solutions to* $\lambda(A^u) \in \partial\Gamma_k$ *are constants. The latter result for* $C^3(\mathbb{R}^n)$ *solutions is independently established by Sheng, Trudinger and Wang in [8]. Our proof is completely different.*

Remark 3 *Writing* $u = w^{-\frac{n-2}{2}}$*, then*

$$A^u \equiv A_w := w\nabla^2 w - \frac{1}{2}|\nabla w|^2 I.$$

Theorem 1, with $\lambda(A^u) \in \partial\Gamma$ *being replaced by* $\lambda(A_w) \in \partial\Gamma$*, holds for* $n = 2$ *as well. See [6].*

In order to illustrate some of the ideas of our proof of Theorem 1 in [6], we give a new proof of the classical Liouville theorem (1). We will derive (1) by using the

Comparison Principle for Δ: *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *containing the origin* 0*. Assume that* $u \in C^2_{loc}(\overline{\Omega} \setminus \{0\})$ *and* $v \in C^2(\overline{\Omega})$ *satisfy*

$$\Delta u \le 0 \quad \text{in } \Omega \setminus \{0\} \qquad \text{and} \qquad \Delta v \ge 0 \quad \text{in } \Omega,$$

and

$$u > v \qquad \text{on } \partial\Omega.$$

Then

$$\inf_{\Omega\setminus\{0\}} (u - v) > 0.$$

It is easy to see from this proof of the Liouville theorem (1) that the following Comparison Principle for locally Lipschitz viscosity solutions of (5), established in [5, 6], is sufficient for a proof of Theorem 1.

Proposition 1 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *containing the origin* 0*, and let* $u \in C^{0,1}_{loc}(\overline{\Omega} \setminus \{0\})$ *and* $v \in C^{0,1}(\overline{\Omega})$*. Assume that* u *and* v *are respectively positive viscosity supersolution and subsolution of (5), and*

$$u > v > 0 \qquad \text{on } \partial\Omega.$$

Then

$$\inf_{\Omega\setminus\{0\}} (u - v) > 0.$$

For the proof of Proposition 1 and Theorem 1, see [5, 6]. In this note, we give the **Proof of Liouville theorem (1) based on the Comparison Principle for Δ.** Let

$$v(x) := \frac{1}{2}[\min_{|y|=1} u(y)]|x|^{2-n}, \quad v_1(x) := \frac{1}{|x|^{n-2}} v(\frac{x}{|x|^2}), \quad u_1(x) := \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}).$$

Since u_1 and v_1 are still harmonic functions, an application of the Comparison Principle for Δ on Ω :=the unit ball yields

$$\liminf_{|y|\to\infty} |y|^{n-2} u(y) > 0. \tag{7}$$

Lemma 1 *For every $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}} u(x + \frac{\lambda^2 (y-x)}{|y-x|^2}) \le u(y) \qquad \forall\ 0 < \lambda < \lambda_0(x), |y-x| \ge \lambda.$$

Proof. Without loss of generality we may take $x = 0$, and we use u_λ to denote $u_{0,\lambda}$. By the positivity and the Lipschitz regularity of u, there exists $r_0 > 0$ such that

$$r^{\frac{n-2}{2}} u(r, \theta) < s^{\frac{n-2}{2}} u(s, \theta), \qquad \forall\ 0 < r < s < r_0,\ \theta \in \mathbb{S}^{n-1}.$$

The above is equivalent to

$$u_\lambda(y) < u(y), \qquad 0 < \lambda < |y| < r_0. \tag{8}$$

We know from (7) that, for some constant $c > 0$,

$$u(y) \ge c|y|^{2-n}, \qquad |y| \ge r_0.$$

Let

$$\lambda_0 := (\frac{c}{\max_{|z|\le r_0} u(z)})^{\frac{1}{n-2}}.$$

Then

$$u_\lambda(y) \le (\frac{\lambda_0}{|y|})^{n-2} (\max_{|z|\le r_0} u(z)) \le c|y|^{2-n} \le u(y), \quad \forall\ 0 < \lambda < \lambda_0, |y| \ge r_0. \tag{9}$$

It follows from (8) and (9) that

$$u_\lambda(y) \le u(y), \quad \forall\ 0 < \lambda < \lambda_0, |y| \ge \lambda.$$

Lemma 1 is established.

$\square$

Because of Lemma 1, we may define, for any $x \in \mathbb{R}^n$ and any $0 < \delta < 1$, that

$$\bar{\lambda}_\delta(x) := \sup\{\mu > 0 \mid u_{x,\lambda}(y) \le (1+\delta) u(y),\ \forall\ 0 < \lambda < \mu, |y-x| \ge \lambda\} \in (0, \infty].$$

Lemma 2 *For any $x \in \mathbb{R}^n$ and any $0 < \delta < 1$, $\bar{\lambda}_\delta(x) = \infty$.*

Proof. We prove it by contradiction. Suppose the contrary, then, for some $x \in \mathbb{R}^n$ and some $0 < \delta < 1$, $\bar{\lambda}_\delta(x) < \infty$. We may assume, without loss of generality, that $x = 0$, and we use u_λ and $\bar{\lambda}_\delta$ to denote respectively $u_{0,\lambda}$ and $\bar{\lambda}_\delta(0)$. Since the harmonicity is invariant under conformal transformations and multiplication by constants, and since

$$u(y) = u_{\bar{\lambda}_\delta}(y) < (1+\delta)u_{\bar{\lambda}_\delta}(y), \qquad \forall\ |y| = \bar{\lambda}_\delta,$$

an application of (7) yields, using the fact that $(u_\lambda)_\lambda \equiv u$,

$$\inf_{0<|y|<\bar{\lambda}_\delta} [(1+\delta)u_{\bar{\lambda}_\delta}(y) - u(y)] > 0.$$

Namely, for some constant $c > 0$,

$$(1+\delta)u(y) - u_{\bar{\lambda}_\delta}(y) \geq c|y|^{2-n}, \qquad \forall\ |y| \geq \bar{\lambda}_\delta. \tag{10}$$

By the uniform continuity of u on the ball $\{z \mid |z| < \bar{\lambda}_\delta\}$, there exists $0 < \epsilon < \bar{\lambda}_\delta$ such that for all $\bar{\lambda}_\delta \leq \lambda \leq \bar{\lambda}_\delta + \epsilon$, and for all $|y| \geq \lambda$, we have

$$\begin{aligned}(1+\delta)u(y) - u_\lambda(y) &\geq (1+\delta)u(y) - u_{\bar{\lambda}_\delta}(y) + [u_{\bar{\lambda}_\delta}(y) - u_\lambda(y)] \\ &\geq c|y|^{2-n} - |y|^{2-n}|\lambda^{n-2}u(\frac{\lambda^2 y}{|y|^2}) - \bar{\lambda}_\delta^{n-2}u(\frac{\bar{\lambda}_\delta^2 y}{|y|^2})| \geq \frac{c}{2}|y|^{2-n}.\end{aligned}$$

This violates the definition of $\bar{\lambda}_\delta$. Lemma 2 is established.

□

By Lemma 2, $\bar{\lambda}_\delta \equiv \infty$ for all $0 < \delta < 1$. Namely,

$$(1+\delta)u(y) \geq u_{x,\lambda}(y), \quad \forall\ 0 < \delta < 1, x \in \mathbb{R}^n, |y - x| \geq \lambda > 0.$$

Sending δ to 0 in the above leads to

$$u(y) \geq u_{x,\lambda}(y), \quad \forall\ x \in \mathbb{R}^n, |y - x| \geq \lambda > 0.$$

This easily implies $u \equiv u(0)$. Liouville theorem (1) is established.

□

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